Decomposition of compact exceptional Lie groups into their maximal tori

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Abstract. In this paper we treat the intersection of fixed point subgroups by the involutive automorphisms of exceptional Lie group $G = F_4,\ E_6, E_7$. We shall find involutive automorphisms of G such that the connected component of the intersection of those fixed point subgroups coincides with the maximal torus of G.

1. Introduction

It is known that the involutive automorphisms of the compact Lie groups play an important role in the theory of symmetric space (c.f. Berger [1]). In [8],[9] Yokota showed that the exceptional symmetric spaces G/H are realized definitely by calculating the fixed point subgroup of the involutive automorphisms $\tilde{\gamma}, \tilde{\gamma}', \tilde{\sigma}, \tilde{\sigma}', \tilde{\iota}$ of G, where $\tilde{\gamma}, \tilde{\gamma}', \tilde{\sigma}, \tilde{\sigma}'$ are induced by $\boldsymbol{R}$-linear transformations $\gamma, \gamma', \sigma, \sigma'$ of $\mathfrak{J}$ and $\tilde{\iota}$ is induced by C-linear transformation ι of $\mathfrak{P}^C$. Here $\gamma, \gamma' \in G_2 \subset F_4 \subset E_6 \subset E_7, \sigma, \sigma' \in F_4 \subset E_6 \subset E_7$ and $\iota \in E_7$. For the cases of the graded Lie algebras $\mathfrak{g}$ of the second kind and third kind, the corresponding subalgebras $\mathfrak{g}_0,\ \mathfrak{g}_{ev},\ \mathfrak{g}_{ed}$ of $\mathfrak{g}$ are realized as the intersection of those fixed point subgroups of the commutative involutive automorphisms ([3],[6],[7],[10],[11],[12]).

In [2],[4],[5] we determined the intersection of those fixed point subgroups of the involutive automorphisms of G when G is a compact exceptional Lie group. We remark that those intersection subgroups are maximal rank of G.

In general, let G be a connected compact Lie group and $\sigma_1, \sigma_2, \cdots, \sigma_m$ commutative automorphism elements of G. Set $G^{\sigma_1,\sigma_2,\cdots,\sigma_k} = \{\alpha \in G \,|\, \sigma_i\alpha = \alpha\sigma_i, i = 1, \cdots, k\}$. We expect that the group $G^{\sigma_1,\sigma_2,\cdots,\sigma_k}$ is a maximal rank subgroup of G. Consider the following degreasing sequence of subgroups of G:

$$G^{\sigma_1} \supset G^{\sigma_1,\sigma_2} \supset \cdots \supset G^{\sigma_1,\cdots,\sigma_m}.$$

Let T^l be the maximal tours of G. In this paper we would like to find $\sigma_1, \sigma_2, \cdots, \sigma_m$ such that the connected component subgroup $(G^{\sigma_1,\sigma_2,\cdots\sigma_k})_0$ of the group $G^{\sigma_1,\sigma_2,\cdots\sigma_k}$

is isomorphic to T^l when G is simply connected compact exceptional Lie groups G_2, F_4, E_6 or E_7. For the case $G = G_2$, we prove that the group $((G_2)^{\gamma,\gamma'})_0 \cong T^2$ by [5], Theorem 1.1.3. Then we shall prove the following:

$$\begin{aligned}
&(1) \quad ((F_4)^{\gamma,\gamma',\sigma,\sigma'})_0 \cong T^4,\\
&(2) \quad ((E_6)^{\gamma,\gamma',\sigma,\sigma'})_0 \cong T^6,\\
&(3) \quad ((E_7)^{\gamma,\gamma',\sigma,\sigma',\iota})_0 \cong T^7.
\end{aligned}$$

For the case $G = E_8$, we conjecture that the group $((E_8)^{\gamma,\gamma',\sigma,\sigma',\upsilon_3})_0 \cong T^8$, where $\lambda' \in E_8$ (As for υ_3, see [3]).

2. Group F_4

The simply connected compact Lie group F_4 is given by the automorphism group of the exceptional Freudenthal algebra $\mathfrak{J}$:

$$F_4 = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{J}) \,|\, \alpha(X \times Y) = \alpha X \times \alpha Y\}.$$

We shall review the definitions of $\boldsymbol{R}$-linear transformations $\gamma, \gamma', \sigma, \sigma'$ of $\mathfrak{J}$([8], [10], [12]).

Firstly we define $\boldsymbol{R}$-linear transformations γ, γ' and γ_1 of $\mathfrak{J}_{\boldsymbol{C}} \oplus M(3, \boldsymbol{C}) = \mathfrak{J}$ by

$$\begin{aligned}
\gamma(X + M) &= X + \gamma(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) = X + (\gamma\boldsymbol{m}_1, \gamma\boldsymbol{m}_2, \gamma\boldsymbol{m}_3),\\
\gamma'(X + M) &= X + \gamma'(\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) = X + (\gamma'\boldsymbol{m}_1, \gamma'\boldsymbol{m}_2, \gamma'\boldsymbol{m}_3),\\
\gamma_1(X + M) &= \overline{X} + \overline{M}, \quad X + M \in \mathfrak{J}_{\boldsymbol{C}} \oplus M(3, \boldsymbol{C}) = \mathfrak{J},
\end{aligned}$$

respectively, where $\mathfrak{J}_{\boldsymbol{C}} = \{X \in M(3, \boldsymbol{C}) \,|\, X^* = X\}$, the right-hand side transformations $\gamma, \gamma' : \boldsymbol{C}^3 \to \boldsymbol{C}^3$ are defined by

$$\gamma\Big(\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}\Big) = \begin{pmatrix} n_1 \\ -n_2 \\ -n_3 \end{pmatrix}, \quad \gamma'\Big(\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}\Big) = \begin{pmatrix} -n_1 \\ n_2 \\ -n_3 \end{pmatrix}, \ n_i \in \boldsymbol{C}.$$

Then $\gamma, \gamma', \gamma_1 \in G_2 \subset F_4$, and $\gamma^2 = {\gamma'}^2 = {\gamma_1}^2 = 1$.

Further we define $\boldsymbol{R}$-linear transfomations σ and σ' of $\mathfrak{J}_{\boldsymbol{C}} \oplus M(3, \boldsymbol{C}) = \mathfrak{J}$ by

$$\begin{aligned}
\sigma(X + M) &= \sigma X + (\boldsymbol{m}_1, -\boldsymbol{m}_2, -\boldsymbol{m}_3),\\
\sigma'(X + M) &= \sigma' X + (-\boldsymbol{m}_1, -\boldsymbol{m}_2, \boldsymbol{m}_3), \quad X + M \in \mathfrak{J}_{\boldsymbol{C}} \oplus M(3, \boldsymbol{C}) = \mathfrak{J},
\end{aligned}$$

respctively, where the right-hand side transformations $\sigma, \sigma' : \mathfrak{J}_{\boldsymbol{C}} \to \mathfrak{J}_{\boldsymbol{C}}$ are defined by

$$\sigma X = \sigma \begin{pmatrix} \xi_1 & x_3 & \overline{x}_2 \\ \overline{x}_3 & \xi_2 & x_1 \\ x_2 & \overline{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x}_2 \\ -\overline{x}_3 & \xi_2 & x_1 \\ -x_2 & \overline{x}_1 & \xi_3 \end{pmatrix}, \ \sigma' X = \begin{pmatrix} \xi_1 & x_3 & -\overline{x}_2 \\ \overline{x}_3 & \xi_2 & -x_1 \\ -x_2 & -\overline{x}_1 & \xi_3 \end{pmatrix}.$$

Then $\sigma, \sigma' \in F_4$ and $\sigma^2 = {\sigma'}^2 = 1$.

The group $\boldsymbol{Z}_2 = \{1, \gamma_1\}$ acts on the group $U(1) \times U(1) \times SU(3)$ by

$$\gamma_1(p, q, A) = (\overline{p}, \overline{q}, \overline{A}).$$

Hence the group $\boldsymbol{Z}_2 = \{1, \gamma_1\}$ acts naturally on the group $(U(1) \times U(1) \times SU(3))/\boldsymbol{Z}_3$.

Let $(U(1) \times U(1) \times SU(3)) \cdot \boldsymbol{Z}_2$ be the semi-direct product of those groups under this action.

Hereafter, ω_1 denotes $-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2} e_1 \in \mathfrak{C}$.

Proposition 2.1. $(F_4)^{\gamma,\gamma'} \cong ((U(1) \times U(1) \times SU(3))/\boldsymbol{Z}_3) \cdot \boldsymbol{Z}_2$, $\boldsymbol{Z}_3 = \{(1, 1, E), (\omega_1, \omega_1, \omega_1 E), ({\omega_1}^2, {\omega_1}^2, {\omega_1}^2 E)\}$.

Proof. We define a mapping $\varphi_4 : (U(1) \times U(1) \times SU(3)) \cdot \boldsymbol{Z}_2 \to (F_4)^{\gamma,\gamma'}$ by

$$\begin{aligned} \varphi_4((p, q, A), 1)(X + M) &= AXA^* + D(p, q)MA^*, \\ \varphi_4((p, q, A), \gamma_1)(X + M) &= A\overline{X}A^* + D(p, q)\overline{M}A^*, \\ & \qquad X + M \in \mathfrak{J}_{\boldsymbol{C}} \oplus M(3, \boldsymbol{C}) = \mathfrak{J}, \end{aligned}$$

where $D(p, q) = \mathrm{diag}(p, q, \overline{pq}) \in SU(3)$. Then φ_4 induces the required isomorphism (see [5] for details). □

Lemma 2.2. *The mapping* $\varphi_4 : (U(1) \times U(1) \times SU(3)) \cdot \boldsymbol{Z}_2 \to (F_4)^{\gamma,\gamma'}$ *satisfies*

$$\sigma = \varphi_4((1, 1, E_{1,-1}), 1), \quad \sigma' = \varphi_4((1, 1, E_{-1,1}), 1),$$

where $E_{1,-1} = \mathrm{diag}(1, -1, -1)$, $E_{-1,1} = \mathrm{diag}(-1, -1, 1) \in SU(3)$.

We denote $U(1) \times \cdots \times U(1), (1, \cdots 1)$ and $(\omega_k, \cdots \omega_k)$ (l-times) by $U(1)^{\times l}, (1)^{\times l}$ and $(\omega_k)^{\times l}$, respectively.

Now, we determine the structures of the group $(F_4)^{\gamma,\gamma',\sigma,\sigma'} = ((F_4)^{\gamma,\gamma'})^{\sigma,\sigma'}$.

Theorem 2.3. $$((F_4)^{\gamma,\gamma',\sigma,\sigma'})_0 \cong U(1)^{\times 4}.$$

Proof. For $\alpha \in (F_4)^{\gamma,\gamma',\sigma,\sigma'} \subset (F_4)^{\gamma,\gamma'}$, there exist $p, q \in U(1)$ and $A \in SU(3)$ such that $\alpha = \varphi_4((p, q, A), 1)$ or $\alpha = \varphi_4((p, q, A), \gamma_1)$ (Proposition 2.1). For the case of $\alpha = \varphi_4((p, q, A), 1)$, by combining the conditions of $\sigma\alpha\sigma = \alpha$ and $\sigma'\alpha\sigma' = \alpha$ with Lemma 2.2, we have

$$\varphi_4((p, q, E_{1,-1}AE_{1,-1}), 1) = \varphi_4((p, q, A), 1)$$

and

$$\varphi_4((p, q, E_{-1,1}AE_{-1,1}), 1) = \varphi_4((p, q, A), 1).$$

Hence

$$\text{(i) } E_{1,-1}AE_{1,-1} = A, \quad \text{(ii)} \begin{cases} p = \omega_1 p \\ q = \omega_1 q \\ E_{1,-1}AE_{1,-1} = \omega_1 A, \end{cases} \qquad \text{(iii)} \begin{cases} p = {\omega_1}^2 p \\ q = {\omega_1}^2 q \\ E_{1,-1}AE_{1,-1} = {\omega_1}^2 A \end{cases}$$

and

$$\text{(iv) } E_{-1,1}AE_{-1,1} = A, \quad \text{(v)} \begin{cases} p = \omega_1 p \\ q = \omega_1 q \\ E_{-1,1}AE_{-1,1} = \omega_1 A, \end{cases} \quad \text{(vi)} \begin{cases} p = {\omega_1}^2 p \\ q = {\omega_1}^2 q \\ E_{-1,1}AE_{-1,1} = {\omega_1}^2 A. \end{cases}$$

We can eliminate the case (ii), (iii), (v) or (vi) because $p \neq 0$ or $q \neq 0$. Hence we have $p, q \in U(1)$ and $A \in S(U(1) \times U(1) \times U(1))$. Since the mapping $U(1) \times U(1) \to S(U(1) \times U(1) \times U(1))$,

$$h(a_1, a_2) = (a_1, a_2, \overline{a_1 a_2})$$

is an isomorphism, the group satisfying with the conditions of case (i) and (iv) is $(U(1)^{\times 4})/\boldsymbol{Z}_3$. For the case of $\alpha = \varphi_4((p, q, A), \gamma_1)$, from $\varphi_4((p, q, A), \gamma_1) = \varphi_4((p, q, A), 1)\gamma_1$, $\varphi_4((1, 1, E_{1,-1}), 1)\gamma_1 = \gamma_1 \varphi_4((1, 1, E_{1,-1}), 1)$ and $\varphi_4((1, 1, E_{-1,1}), 1)\gamma_1 = \gamma_1 \varphi_4((1, 1, E_{-1,1}), 1)$, this case is in the same situation as above. Thus we have $(F_4)^{\gamma,\gamma',\sigma,\sigma'} \cong ((U(1)^{\times 4})/\boldsymbol{Z}_3) \cdot \boldsymbol{Z}_2, \boldsymbol{Z}_3 = \{(1)^{\times 4}, (w_1)^{\times 4}, ({w_1}^2)^{\times 4}\}$. The group $(U(1)^{\times 4})/\boldsymbol{Z}_3$ is naturally isomorphic to the torus $U(1)^{\times 4}$, hence we obtain $(F_4)^{\gamma,\gamma',\sigma,\sigma'} \cong (U(1)^{\times 4}) \cdot \boldsymbol{Z}_2$. Therefore we have the required isomorphism of the theorem. □

3. The group E_6

The simply connected compact Lie group E_6 is given by

$$E_6 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \alpha X \times \alpha Y = \tau\alpha\tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}.$$

$\boldsymbol{R}$-linear transformations $\gamma, \gamma', \gamma_1, \sigma$ and σ' of $\mathfrak{J} = \mathfrak{J}_{\boldsymbol{C}} \oplus M(3, \boldsymbol{C})$ are naturally extended to the C-linear transformations of $\gamma, \gamma', \gamma_1, \sigma$ and σ' of $\mathfrak{J}^C = (\mathfrak{J}_{\boldsymbol{C}})^C \oplus M(3, \boldsymbol{C})^C$. Then we have $\gamma, \gamma', \gamma_1, \sigma, \sigma' \in E_6$.

The group $\boldsymbol{Z}_2 = \{1, \gamma_1\}$ acts on the group $U(1) \times U(1) \times SU(3) \times SU(3)$ by

$$\gamma_1(p, q, A, B) = (\overline{p}, \overline{q}, \overline{B}, \overline{A}).$$

Hence the group $\boldsymbol{Z}_2 = \{1, \gamma_1\}$ acts naturally on the group $(U(1) \times U(1) \times SU(3) \times SU(3))/\boldsymbol{Z}_3$.

Let $(U(1) \times U(1) \times SU(3) \times SU(3)) \cdot \boldsymbol{Z}_2$ be the semi-direct product of those groups under this action.

Proposition 3.1. $(E_6)^{\gamma,\gamma'} \cong ((U(1) \times U(1) \times SU(3) \times SU(3))/\boldsymbol{Z}_3) \cdot \boldsymbol{Z}_2, \boldsymbol{Z}_3 = \{(1, 1, E, E), (\omega_1, \omega_1, \omega_1 E, \omega_1 E), ({\omega_1}^2, {\omega_1}^2, {\omega_1}^2 E, {\omega_1}^2 E)\}$.

Proof. We define a mapping $\varphi_6 : (U(1) \times U(1) \times SU(3) \times SU(3)) \cdot \boldsymbol{Z}_2 \to (E_6)^{\gamma,\gamma'}$ by

$$\begin{aligned} &\varphi_6((p, q, A, B), 1)(X + M) = h(A, B)Xh(A, B)^* + D(p, q)M\tau h(A, B)^*, \\ &\varphi_6((p, q, A, B), \gamma_1)(X + M) = h(A, B)\overline{X}h(A, B)^* + D(p, q)\overline{M}\tau h(A, B)^*, \\ &\qquad\qquad\qquad\qquad X + M \in (\mathfrak{J}_{\boldsymbol{C}})^C \oplus M(3, \boldsymbol{C})^C = \mathfrak{J}^C. \end{aligned}$$

Here $D(p,q) = \mathrm{diag}(p,q,\overline{pq}) \in SU(3)$ and $h : M(3,\boldsymbol{C}) \times M(3,\boldsymbol{C}) \to M(6,\boldsymbol{C})^C$ is defined by

$$h(A,B) = \frac{A+B}{2} + i\frac{A-B}{2}e_1.$$

Then φ_6 induces the required isomorphism (see [5] for details). □

LEMMA 3.2. *The mapping* $\varphi_6 : (U(1) \times U(1) \times SU(3) \times SU(3)) \cdot \boldsymbol{Z}_2 \to (E_6)^{\gamma,\gamma'}$ *satisfies*

$$\sigma = \varphi_6((1,1,E_{1,-1},E_{1,-1}),1), \quad \sigma' = \varphi_6((1,1,E_{-1,1},E_{-1,1}),1).$$

The group $\boldsymbol{Z}_2 = \{1,\gamma_1\}$ acts on the group $U(1)^{\times 6}$ by

$$\gamma_1(p,q,a_1,a_2,a_3,a_4) = (\overline{p},\overline{q},\overline{a}_3,\overline{a}_4,\overline{a}_1,\overline{a}_2).$$

Let $(U(1)^{\times 6}) \cdot \boldsymbol{Z}_2$ be the semi-direct product of those groups under this action.

Now, we determine the structures of the gruop $(E_6)^{\gamma,\gamma',\sigma,\sigma'} = ((E_6)^{\gamma,\gamma'})^{\sigma,\sigma'}$.

THEOREM 3.3. $$((E_6)^{\gamma,\gamma',\sigma,\sigma'})_0 \cong U(1)^{\times 6}.$$

PROOF. For $\alpha \in (E_6)^{\gamma,\gamma',\sigma,\sigma'} \subset (E_6)^{\gamma,\gamma'}$, there exist $p,q \in U(1)$ and $A,B \in SU(6)$ such that $\alpha = \varphi_6((p,q,A,B),1)$ or $\alpha = \varphi_6((p,q,A,B),\gamma_1)$ (Proposition 3.1). For the case of $\alpha = \varphi_6((p,q,A,B),1)$, by combining the conditions $\sigma\alpha\sigma = \alpha$ and $\sigma'\alpha\sigma' = \alpha$ with Lemma 3.2, we have

$$\varphi_6((p,q,E_{1,-1}AE_{1,-1},E_{1,-1}BE_{1,-1}),1) = \varphi_6((p,q,A,B),1)$$

and

$$\varphi_6((p,q,E_{-1,1}AE_{-1,1},E_{-1,1}BE_{-1,1}),1) = \varphi_6((p,q,A,B),1).$$

Hence

$$\text{(i)} \begin{cases} E_{1,-1}AE_{1,-1} = A \\ E_{1,-1}BE_{1,-1} = B, \end{cases} \quad \text{(ii)} \begin{cases} p = \omega_1 p \\ q = \omega_1 q \\ E_{1,-1}AE_{1,-1} = \omega_1 A \\ E_{1,-1}BE_{1,-1} = \omega_1 B, \end{cases} \quad \text{(iii)} \begin{cases} p = {\omega_1}^2 p \\ q = {\omega_1}^2 q \\ E_{1,-1}AE_{1,-1} = {\omega_1}^2 A \\ E_{1,-1}BE_{1,-1} = {\omega_1}^2 B \end{cases}$$

and

$$\text{(iv)} \begin{cases} E_{-1,1}AE_{-1,1} = A \\ E_{-1,1}BE_{-1,1} = B, \end{cases} \quad \text{(v)} \begin{cases} p = \omega_1 p \\ q = \omega_1 q \\ E_{-1,1}AE_{-1,1} = \omega_1 A \\ E_{-1,1}BE_{-1,1} = \omega_1 B, \end{cases} \quad \text{(vi)} \begin{cases} p = {\omega_1}^2 p \\ q = {\omega_1}^2 q \\ E_{-1,1}AE_{-1,1} = {\omega_1}^2 A \\ E_{-1,1}BE_{-1,1} = {\omega_1}^2 B. \end{cases}$$

We can eliminate the case (ii), (iii), (v) or (vi) because $p \neq 0$ or $q \neq 0$. Thus we have $p,q \in U(1)$ and $A,B \in S(U(1)^{\times 3})$. Since the mapping $U(1)^{\times 4} \to S(U(1)^{\times 5})$,

$$h(a_1,a_2,a_3,a_4) = (a_1,a_2,a_3,a_4,\overline{a_1a_2a_3a_4})$$

is an isomorphism, the group satisfying with the conditions of case (i) and (iv) is $(U(1)^{\times 6})/\boldsymbol{Z}_3$. For the case of $\alpha = \varphi_6((p,q,A,B),\gamma_1)$, from $\varphi_6((p,q,A,B),\gamma_1) = \varphi_6((p,q,A,B),1)\gamma_1$, $\varphi_6((1,1,E_{1,-1},E_{1,-1}),1)\gamma_1 = \gamma_1\varphi_6((1,1,E_{1,-1},E_{1,-1}),1)$ and $\varphi_6((1,1,E_{-1,1},E_{-1,1}),1)\gamma_1 = \gamma_1\varphi_6((1,1,E_{-1,1},E_{-1,1}),1)$, this case is in the same situation as above. Thus we have $(E_6)^{\gamma,\gamma',\sigma,\sigma'} \cong ((U(1)^{\times 6})/\boldsymbol{Z}_3)\cdot\boldsymbol{Z}_2$, $\boldsymbol{Z}_3 = \{(1)^{\times 6},(w_1)^{\times 6},({w_1}^2)^{\times 6}\}$. The group $(U(1)^{\times 6})/\boldsymbol{Z}_3$ is naturally isomorphic to the torus $U(1)^{\times 6}$, hence we obtain $(E_6)^{\gamma,\gamma',\sigma,\sigma'} \cong (U(1)^{\times 6})\cdot\boldsymbol{Z}_2$. Therefore we have the required isomorphism of the theorem. □

4. Group E_7

Let $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$. The simply connected compact Lie group E_7 is given by

$$E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P\times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}.$$

Under the identification $(\mathfrak{P}_{\boldsymbol{C}})^C \oplus (M(3,\boldsymbol{C})^C \oplus M(3,\boldsymbol{C})^C)$ with $\mathfrak{P}^C$: $((X,Y,\xi,\eta),(M,N)) = (X+M, Y+N,\xi,\eta)$, C-linear transformations of $\gamma,\gamma',\gamma_1,\sigma$ and σ' of $\mathfrak{J}^C$ are extended to C-linear transformations of $\mathfrak{P}^C$ as

$$\begin{aligned}
\gamma(X+M,Y+N,\xi,\eta) &= (X+\gamma M, Y+\gamma N,\xi,\eta),\\
\gamma'(X+M,Y+N,\xi,\eta) &= (X+\gamma' M, Y+\gamma' N,\xi,\eta),\\
\gamma_1(X+M,Y+N,\xi,\eta) &= (\overline{X}+\overline{M}, \overline{Y}+\overline{N},\xi,\eta),\\
\sigma(X+M,Y+N,\xi,\eta) &= (\sigma X+\sigma M, \sigma Y+\sigma N,\xi,\eta),\\
\gamma(X+M,Y+N,\xi,\eta) &= (\sigma' X+\sigma' M, \sigma' Y+\sigma' N,\xi,\eta),
\end{aligned}$$

where $\gamma M = \mathrm{diag}(1,-1,-1)M, \gamma' M = \mathrm{diag}(-1,-1,1)M, \sigma M = M\mathrm{diag}(1,-1,-1)$ and $\sigma' M = M\mathrm{diag}(-1,-1,1)$.
Moreover we define a C-linear transformation ι of $\mathfrak{P}^C$ by

$$\iota((X+M,Y+N,\xi,\eta) = (-iX-iM, iY+iN,-i\xi,i\eta).$$

The group $\boldsymbol{Z}_2 = \{1,\gamma_1\}$ acts the group $U(1)\times U(1)\times SU(6)$ by

$$\gamma_1(p,q,A) = (\overline{p},\overline{q},\overline{(\mathrm{Ad}J_3)A)}, \quad J_3 = \begin{pmatrix} 0 & E \\ -E & 0\end{pmatrix}.$$

Hence the group $\boldsymbol{Z}_2 = \{1,\gamma_1\}$ acts naturally on the group $(U(1)\times U(1)\times SU(6))/\boldsymbol{Z}_3$.

Let $(U(1)\times U(1)\times SU(6))\cdot\boldsymbol{Z}_2$ be the semi-direct product of those groups under this action.

Proposition 4.1. $(E_7)^{\gamma,\gamma'} \cong ((U(1)\times U(1)\times SU(6))/\boldsymbol{Z}_3)\cdot\boldsymbol{Z}_2$, $\boldsymbol{Z}_3 = \{(1,1,E),(\omega_1,\omega_1,\omega_1 E),({\omega_1}^2,{\omega_1}^2,{\omega_1}^2 E)\}$.

PROOF. We define a mapping $\varphi_7 : (U(1)\times U(1)\times SU(6))\cdot \boldsymbol{Z}_2 \to (E_7)^{\gamma,\gamma'}$ by

$$\begin{aligned}\varphi_7((p,q,A),1)P &= f^{-1}((D(p,q),A)(fP)),\\ \varphi_7((p,q,A),\gamma_1)P &= f^{-1}((D(p,q),A)(f\gamma_1 P)), \quad P\in\mathfrak{P}^C.\end{aligned}$$

Here $D(p,q)=\mathrm{diag}(p,q,\overline{pq})\in SU(3)$ and the mapping f is defined in [9], Section 2.4. Then φ_7 induces the required isomorphism (see [5] for details). $\square$

LEMMA 4.2. *The mapping* $\varphi_7 : (U(1)\times U(1)\times SU(6))\cdot \boldsymbol{Z}_2 \to (E_7)^{\gamma,\gamma'}$ *satisfies*

$$\sigma=\varphi_7((1,1,F_{1,-1}),1), \sigma'=\varphi_7((1,1,F_{-1,1}),1), \iota=\varphi_7((1,1,F_{e_1}),1)$$

where $F_{1,-1}=\mathrm{diag}(1,-1,-1,1,-1,-1), F_{-1,1}=\mathrm{diag}(-1,-1,1,-1,-1,1), F_{e_1}=\mathrm{diag}(e_1,e_1,e_1,-e_1,-e_1,-e_1)\in SU(6)$.

The group $\boldsymbol{Z}_2=\{1,\gamma_1\}$ acts on the group $U(1)^{\times 7}$ by

$$\gamma_1(p,q,a_1,a_2,a_3,a_4,a_5)=(\overline{p},\overline{q},\overline{a}_4,\overline{a}_5,\overline{a}_1,\overline{a}_2,\overline{a}_3)$$

Let $(U(1)^{\times 7})\cdot\boldsymbol{Z}_2$ be the semi-direct product of those groups under this action.

Now, we determine the structures of the group $(E_7)^{\gamma,\gamma',\sigma,\sigma',\iota}=((E_7)^{\gamma,\gamma'})^{\sigma,\sigma',\iota}$.

THEOREM 4.3. $$((E_7)^{\gamma,\gamma',\sigma,\sigma',\iota})_0\cong U(1)^{\times 7}.$$

PROOF. For $\alpha\in(E_7)^{\gamma,\gamma',\sigma,\sigma',\iota}\subset(E_7)^{\gamma,\gamma'}$, there exist $p,q\in U(1)$ and $A\in SU(6)$ such that $\alpha=\varphi_7((p,q,A),1)$ or $\alpha=\varphi_7((p,q,A),\gamma_1)$ (Proposition 4.1). For the case of $\alpha=\varphi_7((p,q,A),1)$, by combining the conditions $\sigma\alpha\sigma=\alpha, \sigma'\alpha\sigma'=\alpha$ and $\iota\alpha\iota^{-1}=\alpha$ with Lemma 4.2, we have

$$\varphi_7((p,q,F_{1,-1}AF_{1,-1}),1)=\varphi_7((p,q,A),1), \varphi_7((p,q,F_{-1,1}AF_{-1,1}),1)=\varphi_7((p,q,A),1)$$

and

$$\varphi_7((p,q,F_{e_1}A{F_{e_1}}^{-1}),1)=\varphi_7((p,q,A),1).$$

Hence

$$\text{(i) } F_{1,-1}AF_{1,-1}=A,\quad \text{(ii)}\left\{\begin{array}{l} p=\omega_1 p\\ q=\omega_1 q\\ F_{1,-1}AF_{1,-1}=\omega_1 A,\end{array}\right. \quad \text{(iii)}\left\{\begin{array}{l} p={\omega_1}^2 p\\ q={\omega_1}^2 q\\ F_{1,-1}AF_{1,-1}={\omega_1}^2 A,\end{array}\right.$$

$$\text{(iv) } F_{-1,1}AF_{-1,1}=A,\quad \text{(v)}\left\{\begin{array}{l} p=\omega_1 p\\ q=\omega_1 q\\ F_{-1,1}AF_{-1,1}=\omega_1 A,\end{array}\right. \quad \text{(vi)}\left\{\begin{array}{l} p={\omega_1}^2 p\\ q={\omega_1}^2 q\\ F_{-1,1}AF_{-1,1}={\omega_1}^2 A.\end{array}\right.$$

and

$$\text{(vii) } F_{e_1}A{F_{e_1}}^{-1}=A,\quad \text{(viii)}\left\{\begin{array}{l} p=\omega_1 p\\ q=\omega_1 q\\ F_{e_1}A{F_{e_1}}^{-1}=\omega_1 A,\end{array}\right. \quad \text{(ix)}\left\{\begin{array}{l} p={\omega_1}^2 p\\ q={\omega_1}^2 q\\ F_{e_1}A{F_{e_1}}^{-1}={\omega_1}^2 A.\end{array}\right.$$

We can eliminate the case (ii), (iii), (v), (vi), (viii) or (ix) because $p \neq 0$ or $q \neq 0$. Thus we have $p, q \in U(1)$ and $A \in S(U(1)^{\times 6})$. Since the mapping $U(1)^{\times 5} \to S(U(1)^{\times 6})$,

$$h(a_1, a_2, a_3, a_4, a_5) = (a_1, a_2, a_3, a_4, a_5, \overline{a_1 a_2 a_3 a_4 a_5})$$

is an isomorphism, the group satisfying with the conditions of case (i),(iv) and (vii) is $(U(1)^{\times 7})/\boldsymbol{Z}_3$. For the case of $\alpha = \varphi_7((p, q, A), \gamma_1)$, from $\varphi_7((p, q, A), \gamma_1) = \varphi_7((p, q, A), 1)\gamma_1, \varphi_7((1, 1, F_{1,-1}), 1)\gamma_1 = \gamma_1\varphi_7((1, 1, F_{1,-1}), 1), \varphi_7((1, 1, F_{-1,1}), 1)\gamma_1 = \gamma_1\varphi_7((1,\ 1, F_{-1,1}), 1)$ and $\varphi_7((1, 1, F_{e_1}), 1)\gamma_1 = \gamma_1\varphi_7((1,\ 1, F_{e_1}), 1)$, this case is in the same situation as above. Thus we have $(E_7)^{\gamma,\gamma',\sigma,\sigma',\iota} \cong ((U(1)^{\times 7})/\boldsymbol{Z}_3) \cdot \boldsymbol{Z}_2$, $\boldsymbol{Z}_3 = \{(1)^{\times 7}, (w_1)^{\times 7}, ({w_1}^2)^{\times 7}\}$. The group $(U(1)^{\times 7})/\boldsymbol{Z}_3$ is naturally isomorphic to the torus $U(1)^{\times 7}$, hence we obtain $(E_7)^{\gamma,\gamma',\sigma,\sigma',\iota} \cong (U(1)^{\times 7}) \cdot \boldsymbol{Z}_2$. Therefore we have the required isomorphism of the theorem. □

3. The group E_8

In the C-vector space ${\mathfrak{e}_8}^C$:

$${\mathfrak{e}_8}^C = {\mathfrak{e}_7}^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C,$$

if we define the Lie bracket $[R_1, R_2]$ by

$$[(\Phi_1, P_1, Q_1, r_1, u_1, v_1), (\Phi_2, P_2, Q_2, r_2, u_2, v_2)] = (\Phi, P, Q, r, u, v),$$

$$\begin{cases} \Phi = [\Phi_1, \Phi_2] + P_1 \times Q_2 - P_2 \times Q_1 \\ P = \Phi_1 P_2 - \Phi_2 P_1 + r_1 P_2 - r_2 P_1 + u_1 Q_2 - u_2 Q_1 \\ Q = \Phi_1 Q_2 - \Phi_2 Q_1 - r_1 Q_2 + r_2 Q_1 + v_1 P_2 - v_2 P_1 \\ r = -\dfrac{1}{8}\{P_1, Q_2\} + \dfrac{1}{8}\{P_2, Q_1\} + u_1 v_2 - u_2 v_1 \\ u = \dfrac{1}{4}\{P_1, P_2\} + 2r_1 u_2 - 2r_2 u_1 \\ v = -\dfrac{1}{4}\{Q_1, Q_2\} - 2r_1 v_2 + 2r_2 v_1, \end{cases}$$

then, ${\mathfrak{e}_8}^C$ becomes a simple C-Lie algebra of type E_8.

The group ${E_8}^C$ is defined to be the automorphism group of the Lie algebra ${\mathfrak{e}_8}^C$:

$${E_8}^C = \{\alpha \in \mathrm{Iso}_C({\mathfrak{e}_8}^C) \,|\, \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}.$$

We define C-linear transformations $\sigma, \sigma', \widetilde{\lambda}$ of ${\mathfrak{e}_8}^C$ respectively by

$$\sigma(\Phi, P, Q, r, u, v) = (\sigma\Phi\sigma, \sigma P, \sigma Q, r, u, v),$$
$$\sigma'(\Phi, P, Q, r, u, v) = (\sigma'\Phi\sigma', \sigma' P, \sigma' Q, r, u, v),$$
$$\widetilde{\lambda}(\Phi, P, Q, r, u, v) = (\lambda\Phi\lambda^{-1}, \lambda Q, -\lambda P, -r, -v, -u),$$

where

$$\sigma\Phi(\phi, A, B, \nu)\sigma = \Phi(\sigma\phi\sigma, \sigma A, \sigma B, \nu),$$
$$\sigma'\Phi(\phi, A, B, \nu)\sigma' = \Phi(\sigma'\phi\sigma', \sigma' A, \sigma' B, \nu),$$
$$\lambda\Phi(\phi, A, B, \nu)\lambda^{-1} = \Phi(-{}^t\phi, -B, -A, -\nu).$$

(σ, σ', λ of the left sides are the same ones used in [3].) Moreover, the complex conjugation in ${\mathfrak e_8}^C$ is denoted by τ:

$$\tau(\Phi, P, Q, r, u, v) = (\tau\Phi\tau, \tau P, \tau Q, \tau r, \tau u, \tau v),$$

where $\tau\Phi(\phi, A, B, \nu)\tau = \Phi(\tau\phi\tau, \tau A, \tau B, \tau\nu)$.

Now, we define the Lie group E_8 as a compact form of the complex Lie group ${E_8}^C$ by

$$E_8 = \{\alpha \in {E_8}^C \mid \tau\widetilde{\lambda}\alpha = \alpha\widetilde{\lambda}\tau\}.$$

Then, E_8 is a simply connected compact simple Lie group of type E_8. Note that $\sigma, \sigma', \widetilde{\lambda} \in E_8$. The Lie algebra $\mathfrak e_8$ of the Lie group E_8 is given by

$$\begin{aligned}\mathfrak e_8 &= \{R \in {\mathfrak e_8}^C \mid \tau\widetilde{\lambda}R = R\} \\ &= \{(\Phi, P, -\tau\lambda P, r, u, -\tau u) \in {\mathfrak e_8}^C \mid \Phi \in \mathfrak e_7, P \in \mathfrak P^C, r \in i\boldsymbol{R}, u \in C\}.\end{aligned}$$

Now, we will investigate the Lie algebra $(\mathfrak e_8)^{\sigma,\sigma'}$ of the group

$$(E_8)^{\sigma,\sigma'} = ((E_8)^{\sigma})^{\sigma'} = (E_8)^{\sigma} \cap (E_8)^{\sigma'}.$$

References

[1] Berger. M., Les espaces symétriques non compacts, Ann. Sci. Ecole Norm. Sup., 74(1957), 85-177.

[2] T. Miyashita, Fixed points subgroups $G^{\sigma,\gamma}$ by two involutive automorphisms σ, γ of compact exceptional Lie groups $G = F_4, E_6$ and E_7, Tsukuba J. Math. 27(2003), 199-215.

[3] T. Miyashita and I. Yokota, 2-graded decompositions of exceptional Lie algebra $\mathfrak g$ and group realizations of $\mathfrak g_{ev}, \mathfrak g_0$, Part III, $G = E_8$, Japanese J. Math. 26(2000), 31-51.

[4] T. Miyashita and I. Yokota, Fixed points subgroups $G^{\sigma,\sigma'}$ by two involutive automorphisms σ, σ' of compact exceptional Lie groups $G = F_4, E_6$ and E_7, Math. J. Toyama Univ. 24(2001),135-149.

[5] T. Miyashita and I. Yokota, Fixed points subgroups $G^{\gamma,\gamma'}$ by two involutive automorphisms γ, γ' of compact exceptional Lie groups $G = G_2, F_4, E_6$ and E_7, Yokohama Math. J., to appear.

[6] T. Miyashita and I. Yokota, 3-graded decompositions of exceptional Lie algebra $\mathfrak g$ and group realizations of $\mathfrak g_{ev}, \mathfrak g_0$ and $\mathfrak g_{ed}$, Part II, $G = E_7$, Part II, Case 1, J. Math. Kyoto Univ., to appear.

[7] T. Miyashita and I. Yokota, 3-graded decompositions of exceptional Lie algebra $\mathfrak g$ and group realizations of $\mathfrak g_{ev}, \mathfrak g_0$ and $\mathfrak g_{ed}$, Part II, $G = E_7$, Part II, Case 2, 3 and 4, J. Math. Kyoto Univ., to appear.

[8] I. Yokota, Realization of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part I, $G = G_2, F_4$, and E_6, Tsukuba J. Math., 4(1990), 185-223.

[9] I. Yokota, Realization of involutive automorphisms σ and G^σ of exceptional linear Lie groups G, Part II, $G = E_7$, Tsukuba J. Math., 4(1990), 378-404.

[10] I. Yokota, 2-graded decompositions of exceptional Lie algebra $\mathfrak g$ and group realizations of $\mathfrak g_{ev}, \mathfrak g_0$, Part I, $G = G_2, F_4, E_6$, Japanese J. Math. 24(1998), 257-296.

[11] I. Yokota, 2-graded decompositions of exceptional Lie algebras $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$, Part II, $G = E_7$, Japanese J. Math. 25(1999), 155-179.
[12] I. Yokota, 3-graded decompositions of exceptional Lie algebra $\mathfrak{g}$ and group realizations of $\mathfrak{g}_{ev}, \mathfrak{g}_0$ and $\mathfrak{g}_{ed}$, Part II, $G = G_2, F_4, E_6$, Part I, J. Math. Kyoto Univ. 41-3(2001), 449-474.

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